

# Numerical evidence for ‘multi-scalar stars’

Scott H. Hawley<sup>1\*</sup> and Matthew W. Choptuik<sup>2†</sup>

<sup>1</sup>Max Planck Institut für Gravitationsphysik, Albert Einstein Institut, D-14476 Golm, Germany and

<sup>2</sup>CIAR Cosmology and Gravity Program, Department of Physics and Astronomy,  
University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z1

We present a class of general relativistic soliton-like solutions composed of multiple minimally coupled, massive, real scalar fields which interact only through the gravitational field. We describe a two-parameter family of solutions we call “phase-shifted boson stars” (parameterized by central density  $\rho_0$  and phase  $\delta$ ), which are obtained by solving the ordinary differential equations associated with boson stars and then altering the phase between the real and imaginary parts of the field. These solutions are similar to boson stars as well as the oscillating soliton stars found by Seidel and Suen [E. Seidel and W. M. Suen, Phys. Rev. Lett. **66**, 1659 (1991)]; in particular, long-time numerical evolutions suggest that phase-shifted boson stars are stable. Our results indicate that scalar soliton-like solutions are perhaps more generic than has been previously thought.

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## I. INTRODUCTION

The nature of dark matter in the universe is currently an open question in physics, with many models being proposed to fill this gap in our understanding, some of which resort to the use of exotic matter. Of interest to us is one class of models composed of massive scalar fields coupled to the general relativistic gravitational field, from which compact, star-like solutions can be formed, solutions which go by the names of “oscillating soliton stars” (or “oscillatons”) [1, 2] for real fields and “boson stars” [3, 4, 5, 6, 7, 8] for complex fields. These star-like solutions have received renewed attention recently, and a substantial body of evidence has been advanced in an effort to show that these fields may be key players on both galactic [9, 10, 11] and cosmological [12] scales. Boson stars have been suggested as alternatives to primordial black holes [13] as well as supermassive black holes in the centers of galaxies [14], and their gravitational lensing properties have been explored [15], further developing the treatment of these solutions as objects of astrophysical interest.

Apart from the possible astrophysical relevance of these star-like objects, we find them interesting to study from a mathematical standpoint as well, for their properties as soliton-like solutions in the nonlinear dynamics of general relativity. The “solution space” of general relativity is still largely unexplored, and these scalar objects comprise simple systems with which to conduct investigations. It is from this viewpoint that we will proceed in this paper.

In 1991, Seidel and Suen [1] considered the model of a real massive scalar field, minimally coupled to the general relativistic gravitational field, with the additional simplifying assumption of spherical symmetry. These au-

thors were interested in the existence of “nontopological solitons” in the model: that is, whether the equations of motion admitted stable, localized, non-singular distributions of matter which could be interpreted as “scalar stars”. A theorem due to Rosen [16] suggested that, should such solutions exist, they could not be static. Thus, Seidel and Suen looked for *periodic* configurations by substituting a particular Fourier ansatz into the equations of motion, and solving the resulting hierarchy of ODEs via a generalized shooting technique. The authors found strong evidence that periodic star-like solutions *did* exist, and, via direct numerical simulation, demonstrated that their “oscillating soliton stars”, if not absolutely stable, had lifetimes many orders of magnitude longer than the stars’ intrinsic dynamical times.<sup>1</sup>

These results were surprising to some researchers, particularly since the model has no conserved Noether current, which, it had been argued, was responsible for the existence of solitonic solutions in other non-linear field theories involving scalar fields [17, 18]. However, at least heuristically, we can understand the existence of the oscillating stars as arising from a balance between the attractive gravitational interaction and the effective repulsive self-interaction generated by the mass of the scalar field (*i.e.* via the dispersion relation of the Klein-Gordon equation).

Recently it was shown by Ureña-López [2] that approximate solutions for boson stars and oscillating soliton stars, or “oscillatons” as he calls them, can both be derived from a single set of equations in a sort of “stationary limit”. The similarities seen between boson stars

\*E-mail address: shawley@aei.mpg.de

†E-mail address: choptuik@physics.ubc.ca

<sup>1</sup> More precisely, the oscillating stars constitute a one-parameter family which may be parametrized by the mean (period-averaged) central density,  $\rho_0$ . As with other relativistic stellar models, a plot of total (ADM) mass versus  $\rho_0$  exhibits a maximum at  $\rho_0^*$ , which seems to coincide with a transition from stable to unstable configurations. As expected, only stars with  $\rho_0 < \rho_0^*$  could be stably evolved for long times.

and oscillating soliton stars in terms of their curves relating mass, radius and central density can thus be related formally.

In this paper, we build on the works of Seidel and Suen and Ureña-López by considering a matter content consisting of *multiple* scalar fields, and we find some further ways in which boson stars and oscillating soliton stars are similar. For the specific case of two scalar fields, we find evidence for a new family of quasi-periodic, solitonic configurations.<sup>2</sup> Together with previous results, this suggests that solitonic solutions are generic to models which couple massive scalar fields through the Einstein gravitational field. We should note, however, that we make no attempt to address the question of whether these multiple scalar fields actually exist in nature; rather we are interested in their existence as valid mathematical solutions in the Einstein-Klein-Gordon system.

We begin by considering  $n$  real, massive Klein-Gordon fields  $\phi_i$ ,  $i = 1, 2, \dots, n$ , without additional self-interaction, minimally coupled to the general relativistic gravitational field. Specifically, choosing units such that  $c = 1$  and  $G = 1$ , the Lagrangian density for the coupled system is

$$\mathcal{L} = \sqrt{-g} \left( R - \sum_{i=1}^n (\phi_i^{;a} \phi_{i;a} - m_i^2 \phi_i^2) \right) \quad (1)$$

where  $g \equiv \det g_{\mu\nu}$ ,  $R$  is the Ricci scalar, and  $m_i$  is the mass of the  $i$ -th scalar field.

We now restrict attention to spherical symmetry, and adopt the “polar/areal” coordinate system, so that the metric takes the form:

$$ds^2 = -\alpha^2(t, r) dt^2 + a^2(t, r) dr^2 + r^2 d\Omega^2 \quad (2)$$

The complete evolution of the scalar fields and spacetime can then be given in terms of a Klein-Gordon equation for each of the  $\phi_i$ , and two constraints derived from the Einstein field equations and the coordinate conditions used to maintain the metric in the form (2). We solve these equations using the same scheme adopted for the critical phenomena study described in [19], and only briefly review that scheme here.

We define the following auxiliary scalar field variables:

$$\Phi_i \equiv \phi_i', \quad \Pi_i \equiv \frac{a}{\alpha} \dot{\phi}_i \quad (3)$$

where all variables are functions of  $t$  and  $r$ ,  $\dot{\cdot} \equiv \partial/\partial t$  and  $' \equiv \partial/\partial r$ . The Klein-Gordon equation is written as the following system:

$$\begin{aligned} \dot{\Pi}_i &= \frac{1}{r^2} \left( \frac{r^2 \alpha}{a} \Pi_i \right)' - m_i^2 \alpha a \phi_i \\ \dot{\Phi}_i &= \left( \frac{\alpha}{a} \Pi_i \right)' \\ \phi_i(t, r) &= \phi_i(t, r_{\max}) - \int_{r_{\max}}^r \Phi(t, \tilde{r}) d\tilde{r} \end{aligned} \quad (4)$$

where  $r = r_{\max}$  is the outer boundary of the computational domain. The constraint equations are the “Hamiltonian constraint”

$$a' = a \frac{1 - a^2}{2r} + 2\pi r a \sum_{i=1}^n (\Pi_i^2 + \Phi_i^2 + a^2 m_i^2 \phi_i^2), \quad (5)$$

and the “slicing condition”

$$\alpha' = \alpha \left( \frac{a^2 - 1}{r} + \frac{a'}{a} - 4\pi r a^2 \sum_{i=1}^n m_i^2 \phi_i^2 \right). \quad (6)$$

For diagnostic purposes, we have also found it useful to compute and monitor the quantities,  $M_i(t, r_{\max})$ , defined by

$$M_i(t, r_{\max}) \equiv 4\pi \int_0^{r_{\max}} \tilde{r}^2 \rho_i(t, \tilde{r}) d\tilde{r}, \quad (7)$$

where

$$\rho_i(t, r) = \frac{\Pi_i^2 + \Phi_i^2 + a^2 \phi_i^2}{2a^2}. \quad (8)$$

Loosely speaking, we can interpret  $M_i(t, r_{\max})$  as the total contribution of field  $i$  to the ADM mass of the spacetime. In particular, so long as no matter out-fluxes through  $r = r_{\max}$ , we have

$$\sum_{i=1}^n M_i(t, r_{\max}) = \text{const.} \quad (9)$$

We solve Eqs. (4)-(6) subject to the boundary conditions  $a(t, 0) = 1$  (local flatness at the origin) and  $a(t, r_{\max}) = 1/a(t, r_{\max})$  (so that  $t$  measures proper time as  $r \rightarrow \infty$ ). As in [19], we use the Sommerfeld condition for a *massless* field to set the values  $\phi(t, r_{\max})$ ,  $\Phi(t, r_{\max})$  and  $\Pi(t, r_{\max})$ . Since the Sommerfeld condition is not ideal for a massive field, we ran our simulations with different values of  $r_{\max}$ , testing for any periodicity or other effect which might be a function of  $r_{\max}$ , and usually ran with an  $r_{\max}$  which was large compared to the time for which we ran the simulation. Even with smaller  $r_{\max}$ , we found our results to be essentially independent of  $r_{\max}$  and attribute this to the fact that there is very little scalar radiation emitted from the soliton-like objects considered here. Our results are also essentially independent of the resolution of the finite differencing algorithm and the Courant-Friedrichs-Levy factor,  $\Delta t/\Delta r$ , and we confirm that our results converge in a second-order-accurate manner using independent residual evaluations.

<sup>2</sup> Although the oscillations appear to be periodic, we do not have a proof of their periodicity and so we use the term “quasi-periodic” to describe their temporal behavior. Small departures from strict periodicity over long times scales are visible in the simulation results; however, these departures become smaller as we decrease the radial mesh spacing  $\Delta r$ , and thus it is plausible that in the limit  $\Delta r \rightarrow 0$ , the solutions are truly periodic.

## II. “PHASE-SHIFTED BOSON STARS”

We start by considering the case  $n = 1$ , so that our matter content is a *single* scalar field,  $\phi(t, r)$ . We note that the Hamiltonian constraint (5) and the slicing condition (6) are unchanged if we trivially decompose  $\phi$  into two identical fields (*i.e.* now choosing  $n = 2$ ), namely  $\phi_1(t, r)$  and  $\phi_2(t, r) = \phi_1(t, r)$  (with  $m_2 = m_1 = 1$ ), such that

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + \phi_2). \quad (10)$$

Further, we note that for fixed  $\alpha(t, r)$  and  $a(t, r)$ , if  $\phi(t, r)$  is a solution of the Klein-Gordon equation (4), then so is  $\kappa \phi(t, r)$  where  $\kappa$  is an arbitrary real constant. Since a soliton solution of the system (4)-(6) is the oscillating soliton star, we see that a trivial multi-scalar soliton solution can be obtained by constructing an oscillating soliton star with a single field  $\phi$  (as described in [1]) and then reinterpreting it as a two-field solution in which  $\phi_1 \equiv \phi_2 \equiv \phi/\sqrt{2}$ .

Moreover, if we wish to model a boson star [3, 4, 5, 6, 7] with no self-interaction potential (often called a “mini-boson star” as in [8]), then we have one massive complex scalar field  $\tilde{\phi}(t, r)$ , for which the real and imaginary parts behave like two real-valued scalar fields:  $\tilde{\phi}(t, r) = \phi_1(t, r) + i\phi_2(t, r)$ . The boson star ansatz is  $\tilde{\phi}(t, r) = \hat{\phi}(r) \exp(\pm i\omega t)$ , where  $\hat{\phi}(r)$  is real. This implies

$$\begin{aligned} \phi_1(t, r) &= \hat{\phi}(r) \cos(\omega t) \\ \phi_2(t, r) &= \hat{\phi}(r) \cos(\omega t + \delta), \end{aligned} \quad (11)$$

where  $\delta = \mp\pi/2$ .

Thus we see that soliton stars and boson stars can both be obtained using two real scalar fields with a constant temporal phase difference. For soliton stars, the fields are identical for all  $r$  and  $t$ ; whereas for boson stars, the fields have identical  $r$ -dependence, and the  $t$ -dependence is the same to within a phase. (In each case, the central density  $\rho_0$  is uniquely fixed by the value of the field at the origin, *e.g.*  $\hat{\phi}(0)$ .)

The work described in the remainder of this paper began in the midst of our numerical evolutions of boson stars [19]. The question arose, “What happens if we solve for the boson star initial data, then ‘manually’ change the phase between the two fields, and then re-solve the constraints to obtain the metric variables?” For future reference, we term such a modified-boson-star configuration a “phase-shifted boson star.” This modification to the boson star data was motivated more by “practical” reasons than “physical” ones — we were interested in studying oscillating soliton stars but found them difficult to construct.

Taking the boson star initial data and setting  $\phi_2(0, r) \equiv \phi_1(0, r)$  resulted in what might be termed a “poor man’s soliton star” (PMSS). In Figures 1, 2 and 3, we show that the resulting solution is very similar to the true soliton star solution, and can perhaps best be regarded as a soliton star with a small perturbation added.

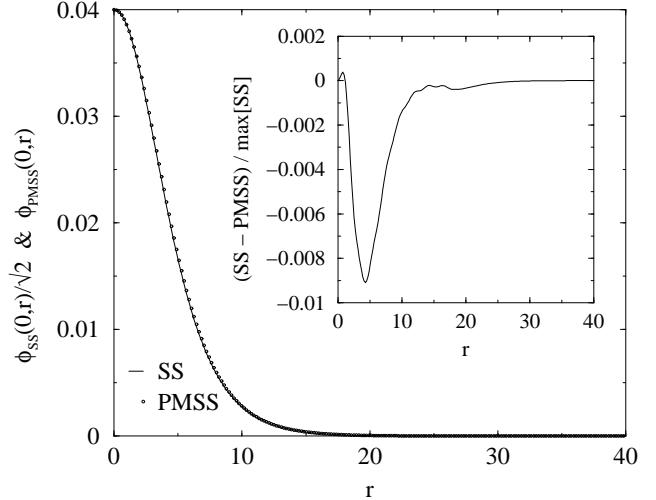


FIG. 1: A comparison between initial data for a true soliton star (SS, solid line) with the “poor man’s soliton star” (PMSS, circles), which is a phase-shifted boson star where the two fields are in phase (*i.e.*,  $\delta = 0$ ), for a particular choice of the central value of the field  $\phi(0, 0)$ . In this figure, we compare the scalar field of the soliton star with one of the two (identical) fields comprising the PMSS, where we have divided the soliton star field by  $\sqrt{2}$  in keeping with the relation (10). The relative difference between the two solutions is plotted in the inset. Given that these two solutions are obtained by solving two rather different sets of ODEs (three simple ODEs for the PMSS, and a complicated system of 10 ODEs for the SS), we find it remarkable that they are so similar. (We note that although we focus on the scalar fields in this figure, the metric variables for the PMSS are also close to those of the SS.)

We then took the boson star initial data  $\hat{\phi}(r)$  and distributed it to  $\phi_1$  and  $\phi_2$  via Eq. (11) using some different value of  $\delta$ , such as  $\delta = -\pi/6$ . (Note that we only apply Eq. (11) for the *initial data*, *i.e.* at  $t = 0$ ; one cannot expect Eq. (11) to describe the fields for all  $t$  if  $\delta \neq \pm\pi/2$ , as the  $\delta = 0$  case demonstrates.) One aspect of the evolution for such a system can be seen in Figure 4.

For each of the many values of  $\delta$  we tried, we found an apparently stable solution which oscillated in some essentially periodic manner for very long times. (The phase was preserved throughout the evolution, *i.e.* it is not the case that the system reverted to a simple “perturbed boson star” over time.) These results led us to conjecture that there may exist a *continuous family* of periodic soliton-like solutions (parameterized by the phase  $\delta$ ) of which our “phase-shifted boson stars” are perturbations. We hope in the future to construct such a family *directly* via a periodic ansatz of the form used by Seidel and Suen for their oscillating soliton stars (with additional terms incorporated to account for the nonlinear coupling between the two scalar fields).

We would like to mention that, for a given  $\hat{\phi}(0)$ , varying  $\delta$  has very little effect on the metric variable  $a(0, r)$ .

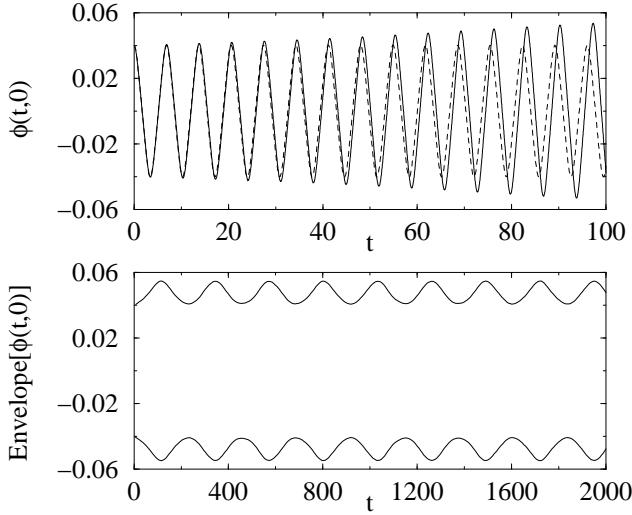


FIG. 2: Central value of the fields  $\phi_1(t, 0) = \phi_2(t, 0)$  vs. time  $t$ , for the “poor man’s soliton star” (PMSS). In the top panel we show both the evolution of the PMSS field (solid line) and, for comparison, a similar evolution for a soliton star field (dashed line). In the lower panel we show only the “envelope” of the oscillations in the PMSS scalar field; the period of variations in the envelope is roughly 32 times the intrinsic period of field. Stable evolutions of this system for  $t > 20000$  have been obtained.

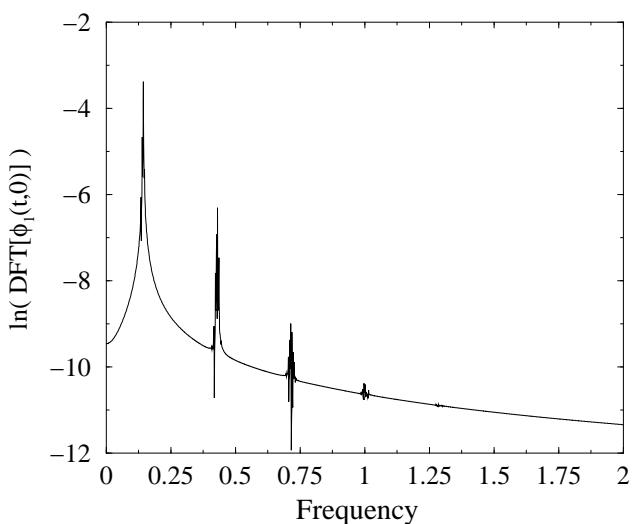


FIG. 3: Fourier transform of the PMSS evolution shown in Figure 2. The spikes in the spectrum correspond closely to the harmonics  $\{\omega, 3\omega, 5\omega, \dots\}$  of the oscillating soliton star. The value of  $\omega$  is an eigenvalue of the soliton star ODE problem; for this simulation,  $\omega \simeq 0.143$ .

Consequently the so-called “stability curves” relating total mass, radius and central density are essentially the same as the curves for boson stars ( $\delta = \pm\pi/2$ ); consequently, we do not consider it relevant to plot them here.

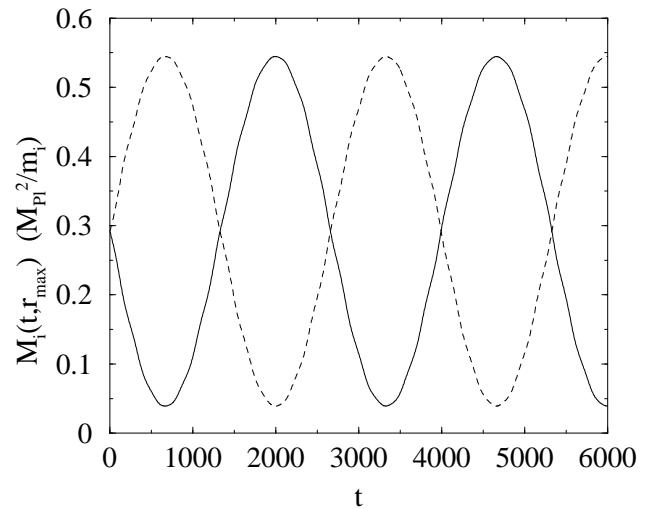


FIG. 4: Time development of the quantities  $M_1(t, r_{\max})$  (solid line) and  $M_2(t, r_{\max})$  (dashed line), defined by (7), for a phase-shifted boson star with  $\delta = -\pi/6$ . We interpret the behaviour seen in this figure as the periodic transfer of significant amounts of energy from one scalar field to the other via the intermediary of the gravitational field. This is reminiscent of “beats” in weakly-coupled harmonic oscillators; in this case, the coupling between the two “oscillators” is gravitational.

### III. CONCLUSIONS

We have strong numerical evidence for the existence of a two-parameter family of soliton-like solutions to the Einstein-Klein-Gordon system (parameterized by central density  $\rho_0$  and phase  $\delta$ ) for the case of two scalar fields. We speculate that extending the system to more scalar fields will yield similar results (especially if one uses a trivial extension like Eq. (10)). The solutions we refer to as “phase-shifted boson stars” consist of boson star initial data for which the phase difference  $\delta$  between the real and imaginary components of the field have been altered. These solutions oscillate in a seemingly periodic manner for very long times; thus they appear to be stable. For the case of  $\delta = 0$ , we obtain close approximations to the oscillating soliton stars of Seidel and Suen. For other values of  $\delta$ , we find solutions which also appear to be stable and periodic; furthermore we can see mass-energy being exchanged between the two fields.

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